Graphing combinations of functions

We've talked about how to graph standard sine and cosine functions in the form

$$a\sin(b(\theta+c))+d$$

$$a\cos(b(\theta+c))+d$$

Now we'll look at how to build combinations of these functions. In other words, if we add, subtract, multiply, or divide two of these functions, what will the graph of the resulting function look like?

Keep in mind that it can be really helpful to graph each of the original functions individually, and then use those two graphs to figure out what the combination graph will look like.

Example

Graph $\cos(3\theta)+\sin(\theta-\pi)$ over the interval $[0,2\pi)$.

We want to recognize that $\cos(3\theta)+\sin(\theta-\pi)$ is really just the combination of $\cos(3\theta)$ and $\sin(\theta-\pi)$. We've added the two functions together to get the combination. So let's look at each of these two functions individually.

If you write $\cos(3\theta)$ in the form $a\cos(b(\theta+c))+d$, you'll get

$$\cos(3\theta)=1\cdot\cos(3(\theta+0))+0$$

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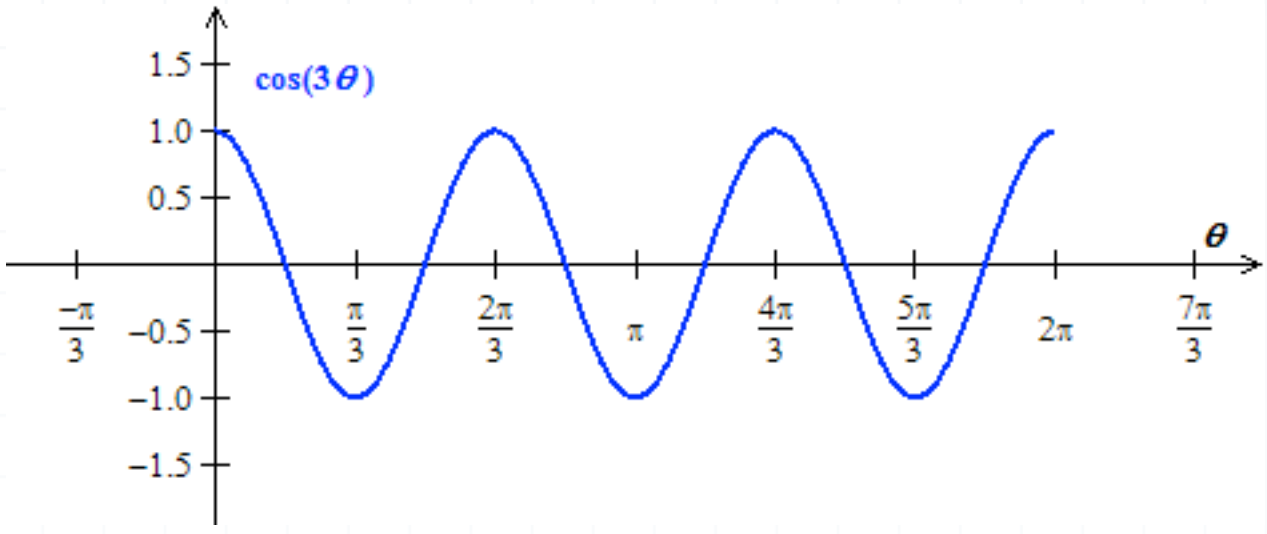
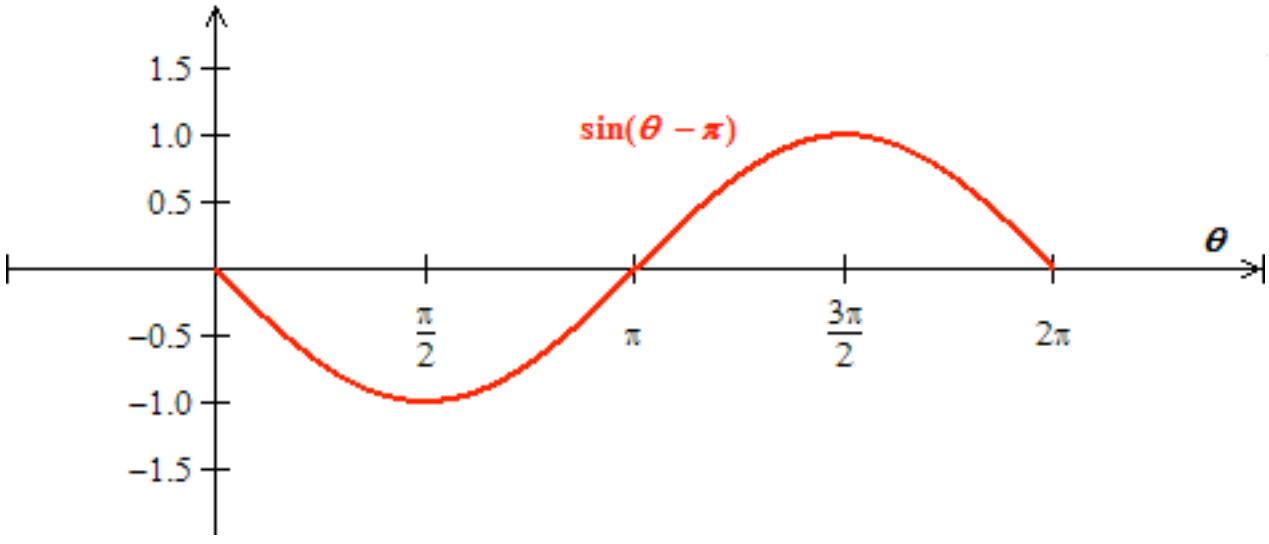
Which means that $a = 1$ (amplitude is 1), $b = 3$ (period is $2\pi/3$), $c = 0$ (there's no horizontal shift), and $d = 0$ (there's no vertical shift). And the graph is



If you write $\sin(\theta - \pi)$ in the form $a\sin(b(\theta + c)) + d$, you'll find that

$$\sin(\theta - \pi) = 1 \cdot \sin(1(\theta - \pi)) + 0$$

so $a = 1$ (amplitude is 1), $b = 1$ (period is 2π), $c = -\pi$ (there's a horizontal shift of $-\pi$ units, which is a shift of π units to the right), and $d = 0$ (there's no vertical shift). And the graph is



The period of $\sin(\theta - \pi)$ is 2π, and the period of $\cos(3\theta)$ is $2\pi/3$, so the period of $\sin(\theta - \pi)$ is three times the period of $\cos(3\theta)$. When the combination function is a sum or difference like it is in this problem, you take the larger

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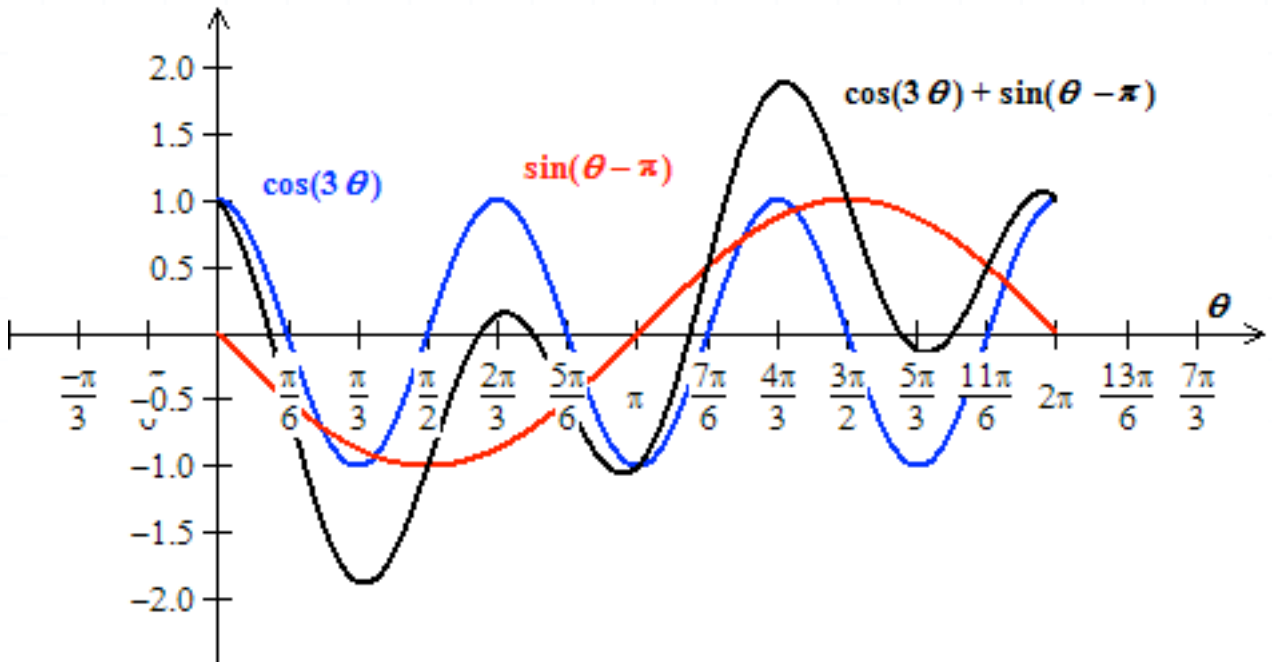
of the two periods of the original functions as the period of the combination. So since 2π is larger than $2\pi/3$, the period of the combination is 2π.

We can sketch the graph of the combination simply by adding together the values of the other two functions at each point. For example, at $\theta = 0$, $\cos(3\theta)$ has a value of 1, and $\sin(\theta - \pi)$ has a value of 0, which means $\cos(3\theta) + \sin(\theta - \pi)$ will have a value of $1 + 0 = 1$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over the interval $[0,2\pi)$.



The difference of two functions, $f - g$, is just the sum of f and $-g$, so let's go on and take a look at the product of a sine function and a cosine function.

Example

Graph $(\sin(2\theta))(2\cos\theta + 1)$ over the interval $[0,2\pi)$.

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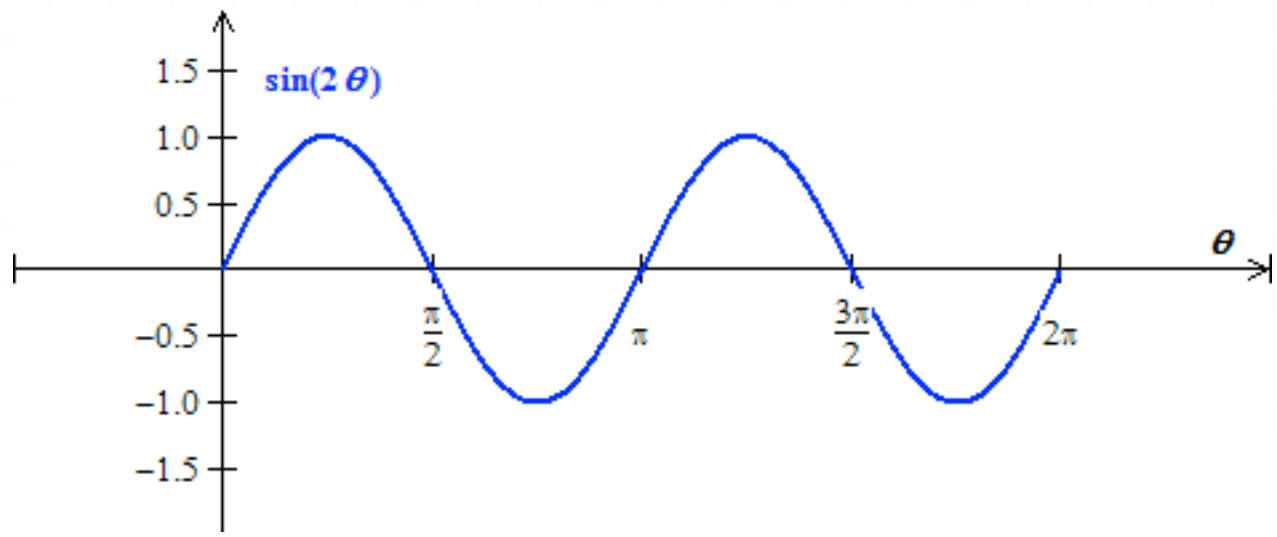
We want to recognize that $(\sin(2\theta))(2\cos\theta + 1)$ is really just the combination of $\sin(2\theta)$ and $2\cos\theta + 1$. We've taken the product of the two functions together to get the combination. So let's look at each of these two functions individually.

You can write $\sin(2\theta)$ as

$$1 \cdot \sin(2(\theta + 0))$$

so $a = 1$ (amplitude is 1), $b = 2$ (period is $2\pi/2 = \pi$), $c = 0$ (there's no horizontal shift), and $d = 0$ (there's no vertical shift). And the graph is



And you can write $2\cos\theta + 1$ as

$$2\cos(1(\theta + 0)) + 1$$

Thus this function has $a = 2$ (amplitude is 2), $b = 1$ (period is 2π), $c = 0$ (there's no horizontal shift), and $d = 1$ (there's a vertical shift upwards by 1 unit). And the graph is

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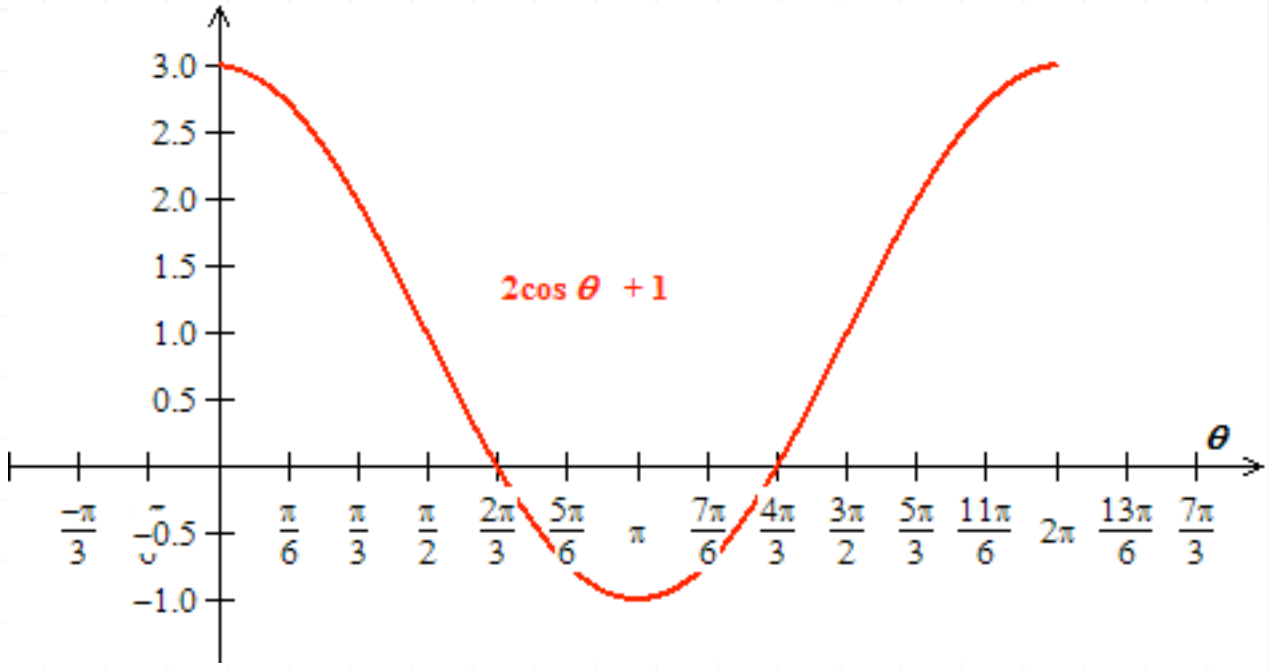

The period of $2\cos\theta + 1$ is 2π, and the period of $\sin(2\theta)$ is π, so the period of $2\cos\theta + 1$ is twice the period of $\sin(2\theta)$. When the combination function is a product or quotient like it is in this problem, you take the larger of the two periods of the original functions as the period of the combination. So since 2π is larger than π, the period of the combination is 2π.

We can sketch the graph of the combination simply by multiplying together the values of the other two functions at each point. For example, at $\theta = 0$, $2\cos\theta + 1$ has a value of 3, and $\sin(2\theta)$ has a value of 0, which means $(\sin(2\theta))(2\cos\theta + 1)$ will have a value of $3(0) = 0$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over the interval $[0,2\pi)$.

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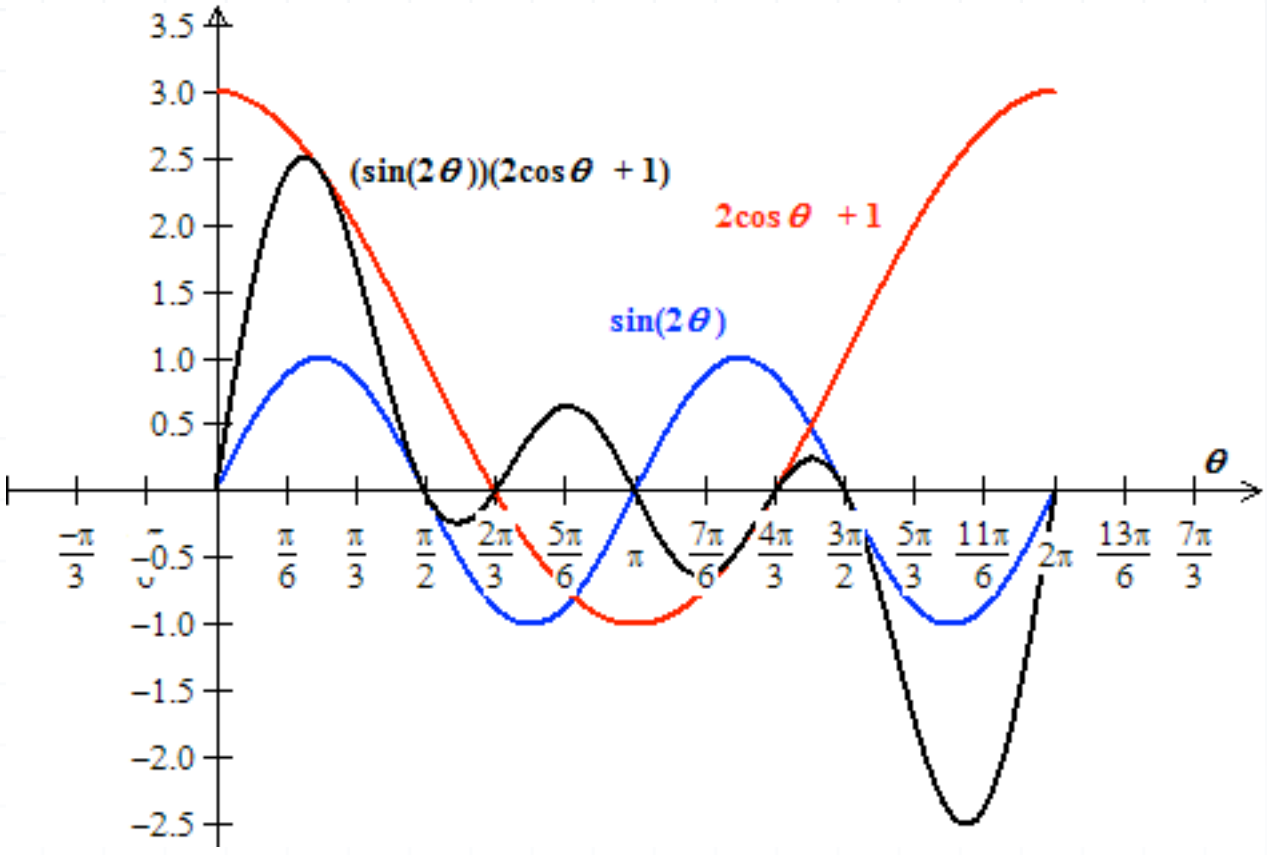

Things get more complicated when we consider a quotient of trig functions, because the quotient is undefined at every angle θ at which the function in the denominator is equal to 0.

Example

Graph the function over the interval $[0,2\pi)$.

$$\frac{\cos\left(\theta+\frac{\pi}{4}\right)}{\sin\left(3\theta-\frac{\pi}{2}\right)}$$

We want to recognize that the given combination is really just the combination of $\cos(\theta+(\pi/4))$ and $\sin(3\theta-(\pi/2))$. We've divided the two

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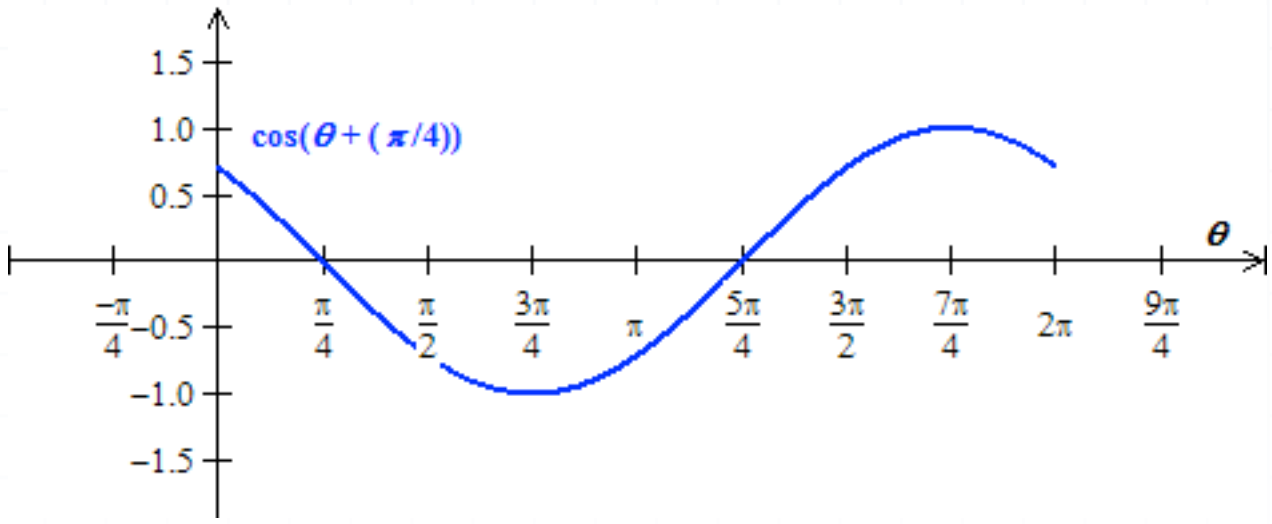
functions to get the combination. So let's look at each of these two functions individually.

The function in the numerator, $\cos(\theta + (\pi/4))$, can be expressed in the form $a\cos(b(\theta + c)) + d$ with $a = 1$ (amplitude is 1), $b = 1$ (period is 2π), $c = \pi/4$ (there's a horizontal shift by $\pi/4$ units, which means the graph is shifted $\pi/4$ units to the left), and $d = 0$ (there's no vertical shift). And the graph is



The function in the denominator, $\sin(3\theta - (\pi/2))$, can be rewritten by factoring 3 out of each term in $3\theta - (\pi/2)$:

$$\sin\left(3\theta - \frac{\pi}{2}\right)$$

$$\sin\left(3\left[\frac{1}{3}(3\theta) - \frac{1}{3}\left(\frac{\pi}{2}\right)\right]\right)$$

$$\sin\left(3\left(\theta - \frac{\pi}{6}\right)\right)$$

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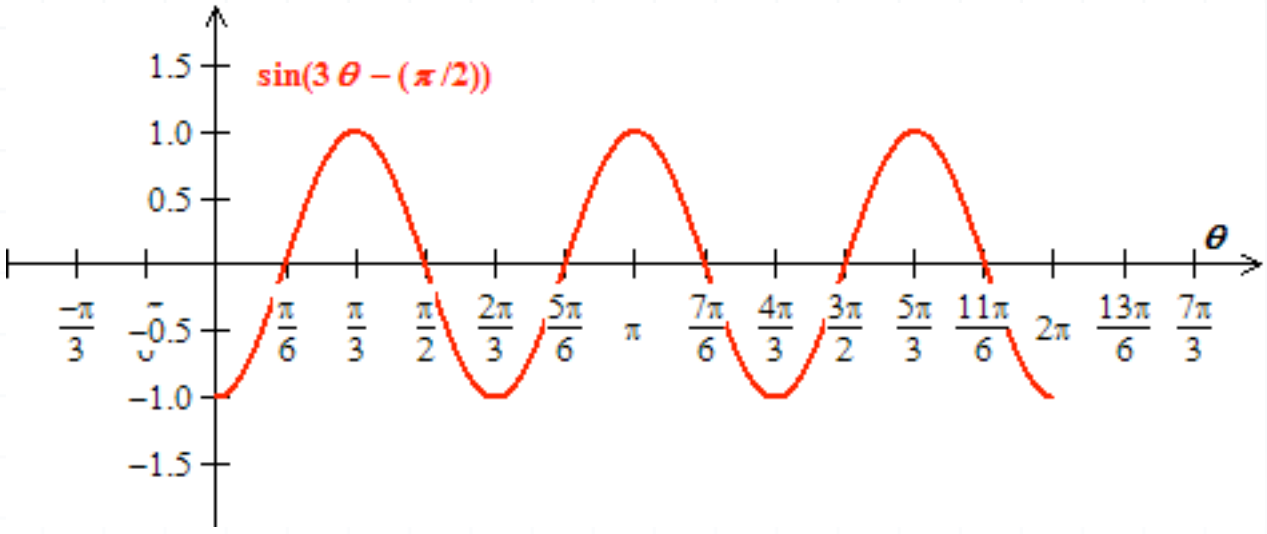
So $a = 1$ (amplitude is 1), $b = 3$ (period is $2\pi/3$), $c = -\pi/6$ (there's a horizontal shift of $-\pi/6$, which means the graph is shifted $\pi/6$ units to the right), and $d = 0$ (there's no vertical shift). And the graph is



We can expect that the behavior of the combination quotient function will be a little bizarre near any values where $\sin(3\theta - (\pi/2))$ gets close to 0, since $\sin(3\theta - (\pi/2))$ is the denominator of the quotient combination, and a function is undefined whenever it's denominator is equal to 0.

Let's look at where $\sin(3\theta - (\pi/2))$ will get close to 0. We'll set $\sin(3\theta - (\pi/2))$ equal to 0 and solve for θ.

$$\sin\left(3\theta - \frac{\pi}{2}\right) = 0$$

The sine function is equal to 0 at angles of 0, π, 2π, 3π, 4π, $\ldots$, in other words, at multiples of π, or any value $n\pi$ where n is an integer. Therefore we can solve

$$3\theta - \frac{\pi}{2} = n\pi$$

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To solve this equation, all that's needed is a series of steps that involve algebraic manipulation.

$$\frac{2(3\theta)-1(\pi)}{2}=n\pi$$

$$\frac{6\theta-\pi}{2}=n\pi$$

$$6\theta-\pi=2n\pi$$

$$6\theta=2n\pi+\pi$$

$$6\theta=(2n+1)\pi$$

$$\theta=\frac{(2n+1)\pi}{6}$$

To see where this is true inside $[0,2\pi)$, we'll solve the inequality

$$0\le\frac{(2n+1)\pi}{6}<2\pi$$

$$0\left(\frac{6}{\pi}\right)\le\left[\frac{(2n+1)\pi}{6}\right]\left(\frac{6}{\pi}\right)<2\pi\left(\frac{6}{\pi}\right)$$

$$0\le 2n+1<12$$

$$-1\le 2n<11$$

$$-\frac{1}{2}\le n<\frac{11}{2}$$

The integers n that lie in the interval

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$$\left[-\frac{1}{2}, \frac{11}{2}\right)$$

are 0, 1, 2, 3, 4, and 5. So we can expect that in the interval $[0,2\pi)$ the graph of the quotient,

$$\frac{\cos\left(\theta + \left(\frac{\pi}{4}\right)\right)}{\sin\left(3\theta - \left(\frac{\pi}{2}\right)\right)}$$

will exhibit "strange" behavior near the angles $\theta = (2n+1)\pi/6$ with $n \in \{0,1,2,3,4,5\}$:

$$n = 0 \quad \rightarrow \quad \theta = [2(0)+1]\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

$$n = 1 \quad \rightarrow \quad \theta = [2(1)+1]\left(\frac{\pi}{6}\right) = \frac{3\pi}{6} = \frac{\pi}{2}$$

$$n = 2 \quad \rightarrow \quad \theta = [2(2)+1]\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$$

$$n = 3 \quad \rightarrow \quad \theta = [2(3)+1]\left(\frac{\pi}{6}\right) = \frac{7\pi}{6}$$

$$n = 4 \quad \rightarrow \quad \theta = [2(4)+1]\left(\frac{\pi}{6}\right) = \frac{9\pi}{6} = \frac{3\pi}{2}$$

$$n = 5 \quad \rightarrow \quad \theta = [2(5)+1]\left(\frac{\pi}{6}\right) = \frac{11\pi}{6}$$

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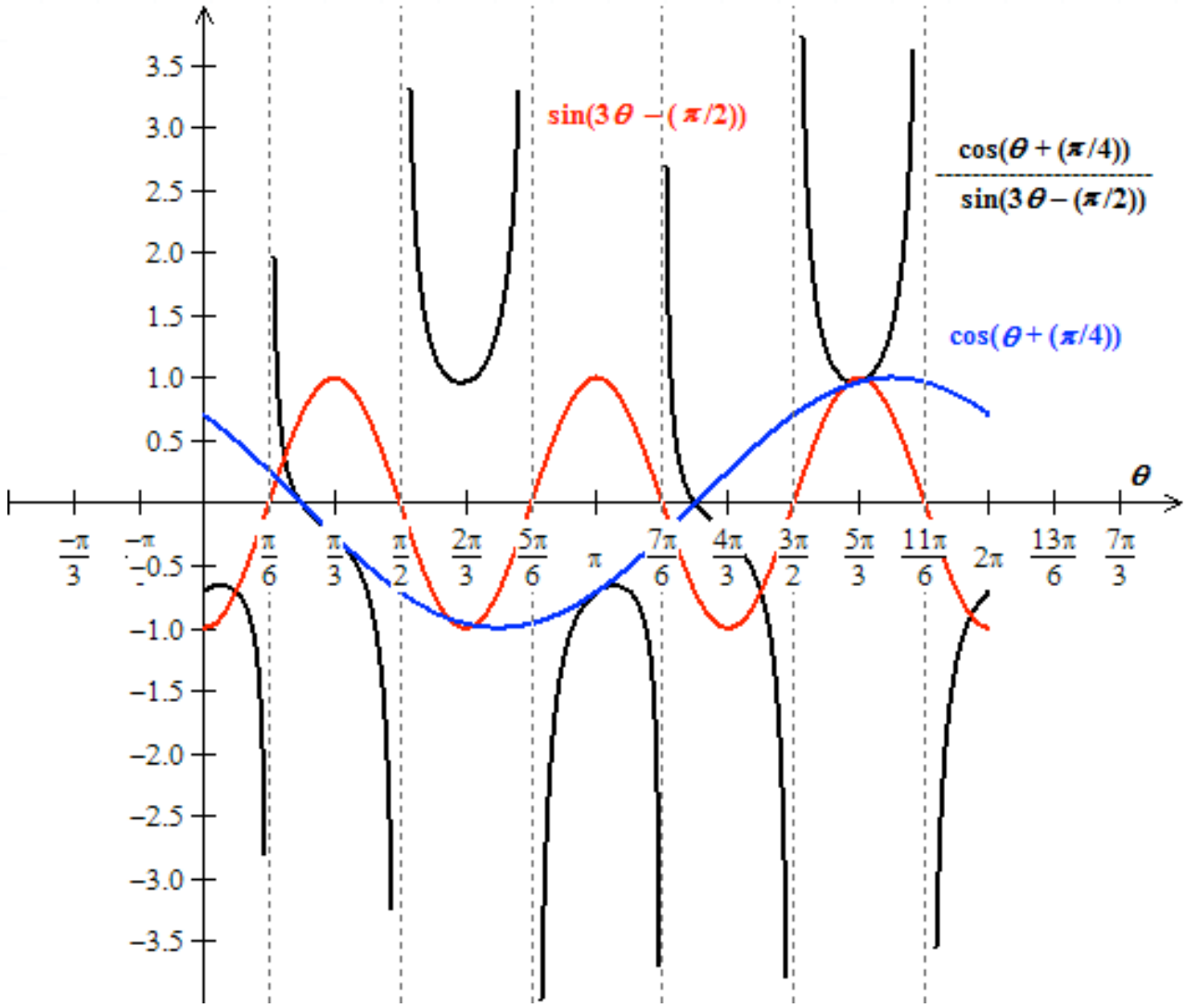
This is exactly what we find when we look at the graph of the quotient. Notice that the value of the quotient

$$\frac{\cos\left(\theta+\left(\frac{\pi}{4}\right)\right)}{\sin\left(3\theta-\left(\frac{\pi}{2}\right)\right)}$$

goes to $+\infty$ as θ approaches $\pi/6$, $\pi/2$, $7\pi/6$, or $3\pi/2$ from the right, and as θ approaches $5\pi/6$ or $11\pi/6$ from the left. Similarly, the value of the quotient goes to $-\infty$ as θ approaches $\pi/6$, $\pi/2$, $7\pi/6$, or $3\pi/2$ from the left, and as θ approaches $5\pi/6$ or $11\pi/6$ from the right.



The period of $\cos(\theta+(\pi/4))$ is 2π, and the period of $\sin(3\theta-(\pi/2))$ is $2\pi/3$, so the period of $\cos(\theta+(\pi/4))$ is three times the period of $\sin(3\theta-(\pi/2))$. When

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the combination function is a product or quotient like it is in this problem, you take the larger of the two periods of the original functions as the period of the combination. So since 2π is larger than $2\pi/3$, the period of the combination is 2π.

We can sketch the graph of the combination simply by dividing the values of the other two functions at each point. For example, at $\theta = 0$, $\cos(\theta + (\pi/4))$ has a value of $\sqrt{2}/2$, and $\sin(3\theta - (\pi/2))$ has a value of -1, which means the combination quotient will have a value of

$$\frac{\sqrt{2}}{2}/(-1) = -\frac{\sqrt{2}}{2}$$

and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over the interval $[0,2\pi)$.

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